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Covering systems in number fields

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ABSTRACT

M. Filaseta, K. Ford, S. Konyagin, C. Pomerance and G. Yu proved that if the reciprocal sum of the moduli of a covering system is bounded, then the least modulus is also bounded, which confirms a conjecture of P. Erdős and J.L. Selfridge. They also showed that, for $K > 1$, the complement in $\mathbb{Z}$ of any union of residue classes $r(n) \pmod n$ with distinct $n \in (N, KN]$ has density at least d_K for N sufficiently large, which implies a conjecture of P. Erdős and R.L. Graham. In this paper, we extend these results to covering systems of the ring of integers of an arbitrary number field $F/\mathbb{Q}$.

1. Introduction

A finite collection of congruence classes, $\{a_1 \pmod{m_1}, \ldots, a_k \pmod{m_k}\}$ with $m_i > 1$ is called a covering system if each integer lies in at least one of them. Furthermore, if a covering system covers every integer exactly once, then it is said to be an exact covering system. A covering system with the least number of distinct moduli is the following:

$$0 \pmod 2, \quad 0 \pmod 3, \quad 1 \pmod 4, \quad 1 \pmod 6, \quad 11 \pmod{12}. \tag{1.1}$$

The concept of a covering system of $\mathbb{Z}$ was first introduced by P. Erdős, who answered Romanoff's question: Can every sufficiently large integer be expressed as the sum of a power of 2 and a prime? He showed that the answer is no, and in his proof [2], he used (1.1) with the last two classes replaced by

$$3 \pmod 8, \quad 7 \pmod{12}, \quad 23 \pmod{24}.$$

Here are some famous conjectures concerning covering systems.

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Least Modulus Problem (Erdős' conjecture). *Can the least modulus in a covering system with distinct moduli be arbitrarily large*?

D.J. Gibson [8] found a covering system with distinct moduli where the least modulus is 25, and a covering system with distinct moduli $\geqslant 40$ has been recently discovered by P. Nielsen [15].

Odd Moduli Problem. *Is there a covering system with distinct odd moduli*?

Such a covering system is called an odd covering. Erdős, convinced that an odd covering exists, offered \$25 for a proof that an odd covering does not exist. In 1970 Selfridge conjectured the opposite and offered money for exhibiting an odd covering (cf. the Introduction of [6]).

Schinzel's Conjecture. *In every covering system, there is a modulus that divides one of the others.*

In [5], J. Fabrykowski and T. Smotzer gave a simple proof showing that if Schinzel's Conjecture is false, then there exists an odd covering.

Recently, some conjectures of P. Erdős, J.L. Selfridge and R.L. Graham were confirmed by M. Filaseta, K. Ford, S. Konyagin, C. Pomerance and G. Yu [7]. Erdős [3] conjectured the following

Conjecture I. *For any number B, there is a number N_B, such that in a covering system with distinct moduli greater than N_B, the sum of reciprocals of these moduli is greater than B.*

It is also interesting to study systems of residue classes where the moduli are distinct and come from an interval $(N, KN]$. Erdős and Graham [4] made a following conjecture.

Conjecture II. *For each number $K > 1$ there is a positive number d_K such that if N is sufficiently large, depending on K, and we choose arbitrary integers $r(n)$ for each $n \in (N, KN]$, then the complement in $\mathbb{Z}$ of the union of the residue classes $r(n) \pmod n$ has density at least d_K.*

In [7], stronger forms of these conjectures were proved.

Let

$$L(N, s) = \exp\left(\log N \frac{\log\log(s\log N)}{\log(s\log N)}\right).$$

Theorem I. *Suppose $0 < b < \frac{1}{2}$, $0 < c < \frac{1}{3}(1 - 4b^2)$ and let N be sufficiently large, depending on the choice of b and c. Suppose C is a finite set of congruence classes with moduli $> N$, each modulus appearing at most s times, where $s \leqslant \exp(b\sqrt{\log N \log\log N})$, and such that*

$$\sum_{(r \bmod n)\in C} \frac{1}{n} \leqslant c \log L(N, s). \tag{1.2}$$

Then C is not a covering system.

Theorem II. *Suppose $0 < \varepsilon < 1/2$, $0 < b < \frac{1}{2}\sqrt{\varepsilon}$ and $N \geqslant 100$. Suppose that C is a finite set of congruence classes with moduli from $(N, KN]$, each modulus appearing at most s times, where $s \leqslant \exp(b\sqrt{\log N \log\log N})$ and $K = L(N, s)^{(1/2-\varepsilon)/s}$. Then the density of the integers not covered by C is*

$$\geqslant \left(1 + O\left(\frac{1}{(\log N)^{\lambda}}\right)\right) \prod_{(r \bmod n)\in C} \left(1 - \frac{1}{n}\right),$$

where λ is a positive constant depending only on ε and b.

It is not hard to see that Theorems I and II imply Conjectures I and II, respectively, by setting $s=1$.

Covering systems of groups by subgroups or cosets of subgroups, which is the most natural generalization of covering systems of $\mathbb{Z}$, have been investigated by B.H. Neumann [14], M.M. Parmenter [16, 17] and Z.W. Sun [19,20]. Also, T. Cochrane and G. Myerson [1] discuss a type of covering system of $\mathbb{Z}^n$, namely system of congruences

$$\sum_{j=1}^{n} a_{ij}x_j \equiv c_i \pmod{m_i}, \quad 1 \leqslant i \leqslant k,$$

where $a_{ij}, c_i \in \mathbb{Z}$ and $(a_{i0}, \ldots, a_{in}, c_i, m_i) = 1$ and such that every n-tuple of integers satisfies at least one of them. Further results on covering systems of $\mathbb{Z}^n$ are given by B. Jin, G. Myerson [11] and A. Schinzel [18].

In this paper, we generalize Theorems I and II to arbitrary number fields. We take advantage of the facts that all the ideals in the ring of integers of a number field have unique factorization into prime ideals, the greatest common divisor and the least common multiple of ideals are defined, and the Chinese Remainder Theorem holds, which are necessary in the proofs of the results from [7].

Now, we introduce the concept of covering systems in number fields. For example, consider the field of Gaussian rationals $\mathbb{Q}(i)$ with ring of integers $\mathbb{Z}[i] = \{a+bi\colon\ a, b \in \mathbb{Z}\}$, which is the set of Gaussian integers. Let $I = (1+i)$, which is an ideal in $\mathbb{Z}[i]$ generated by $1+i$. Then, we can see that $\mathbb{Z}[i] = I \cup \{1+I\}$. In other words, $\{0 \pmod{I}, 1 \pmod{I}\}$ covers $\mathbb{Z}[i]$. We say $\{0 \pmod{I}, 1 \pmod{I}\}$ is a covering system of $\mathbb{Z}[i]$ (or in $\mathbb{Q}(i)$). More generally, let $F/\mathbb{Q}$ be a number field of degree d with ring of integers $\mathcal{O}_F$. We call $\{r_1 \pmod{I_1}, \ldots, r_k \pmod{I_k}\}$ a covering system in F (or of $\mathcal{O}_F$) if for each $i \leqslant k$, $r_i \in \mathcal{O}_F$, I_i is an ideal in $\mathcal{O}_F$, and $\mathcal{O}_F = \bigcup_{i=1}^{k}\{r_i + I_i\}$. Furthermore, if a covering system covers every element of $\mathcal{O}_F$ exactly once, then it is said to be an exact covering system. Thus, in fact, $\{0 \pmod{I}, 1 \pmod{I}\}$ is an exact covering system of $\mathbb{Z}[i]$.

We remark that a covering system in a number field can be identified with a covering system of $\mathbb{Z}^d$ by cosets of subgroups, where d is the degree of the number field, since the ring of integers of a number field with degree d is isomorphic to $\mathbb{Z}^d$ as an additive group. However, the moduli from a covering system in a number field are ideals in the ring of integers. Thus, the covering systems in a number field are more restrictive than those of $\mathbb{Z}^n$ by cosets of any subgroups.

Here, note that if $\{r_1 \pmod{I}, \ldots, r_k \pmod{I}\}$ is an exact covering system, then we must have $k = |\mathcal{O}_F/I| = \|I\|$, which is the norm of I. Analogous to the notion of density for sets of integers, we say that the density of each $r_i \pmod{I}$ is $1/\|I\|$.

In order to deal with ideals in the ring of integers, we also introduce the functions $f(n)$ and $g(n)$, which denote the number of ideals of norm n and the number of prime ideals of norm n, respectively. In particular, we use the following key propositions.

Proposition 1. *(See [13, Corollary of Theorem 39].) Let $F/\mathbb{Q}$ be a number field of degree d. Then,*

$$\sum_{n\leqslant x} f(n) = c_F x + O\left(x^{1-\frac{1}{d}}\right),$$

where c_F is a constant depending on F.

Proposition 2. *(See [12, p. 670].) Let $F/\mathbb{Q}$ be a number field of degree d. Then,*

$$\sum_{n\leqslant x} g(n) = \mathrm{Li}(x) + O\left(x\exp\left(-(\log x)^{1/13}\right)\right), \quad \textit{where } \mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}.$$

Here and throughout the paper, constants implied by the O-symbol may depend on the field F. Dependence on any other quantity will be indicated by a subscript. We remark that a stronger version of Proposition 2 is possible using Theorem 5.33 of [10] (the term involving a possible exceptional zero is absorbed into the error estimate at the cost of the O-constant, which thus depends on the field F in an inexplicit way). We can easily see that $g(p^n) \leqslant d$ and $g(p^n) = 0$ if $n > d$, where p is a prime.

We adopt some notation which is similar to that in [7]. A finite collection of congruence classes $C = \{r_1 \pmod{I_1}, \ldots, r_k \pmod{I_k}\}$ in a number field $F/\mathbb{Q}$ we call a residue system. Let $S(C)$ be the multiset $\{I_1, \ldots, I_k\}$ and we say that the multiplicity of I_i is the number of times that I_i appears in $S(C)$. By $\delta(C)$ we denote the density of the elements of the ring of integers not covered by C, and we also set

$$\alpha(C) = \prod_{I \in S(C)} \left(1 - \frac{1}{\|I\|}\right).$$

The goal of this paper is to derive analogues of all the lemmas and theorems of [7], including Theorems I and II, in the number field setting. In Section 2, we present analogues of many preparatory lemmas from [7]. Most of the proofs are very similar to those of [7]. A notable exception is Lemma 2.6 below. In Section 3, we prove our main theorems, which are analogues of Theorems 2–4 of [7]. Let us state three of our results, the first and the third being analogues of Theorem I and Theorem II, respectively.

Theorem 1. *Suppose $0 < b < \frac{1}{2}$, $0 < c < \frac{1}{3}(1 - 4b^2)$. Let $F/\mathbb{Q}$ be a number field of degree $d \geqslant 1$. Let N be sufficiently large, depending on the choice of b, c and F. Suppose C is a residue system in $F/\mathbb{Q}$ with $S(C)$ consisting of ideals $\|I\| > N$, each having multiplicity at most s, where $s \leqslant \exp(b\sqrt{\log N \log\log N})$, and such that*

$$\sum_{I \in S(C)} \frac{1}{\|I\|} \leqslant c \log L(N, s). \tag{1.3}$$

Then $\delta(C) > 0$.

Theorem 2. *Suppose that C is a residue system of a number field $F/\mathbb{Q}$ of degree d. Suppose $0 < \varepsilon < (1 - \log 2)^{-1}$, $b < \frac{1}{2}\sqrt{(1 - \log 2)\varepsilon}$, N is sufficiently large, depending on the choice of ε, b and F, and $S(C)$ consists of ideals whose norms are in $(N, KN]$ with multiplicity at most s, where $s \leqslant \exp(b\sqrt{\log N \log\log N})$ and $K = L(N, s)^{((1-\log 2)^{-1}-\varepsilon)/c_F s}$. Then $\delta(C) > 0$.*

As in [7], the following theorem shows that if K is a bit smaller than in Theorem 2, then we have

$$\delta(C) \geqslant \big(1 + o(1)\big)\alpha(C).$$

Theorem 3. *Suppose $0 < \varepsilon < 1/2$, $0 < b < \frac{1}{2}\sqrt{\varepsilon}$ and $N \geqslant 100$. Suppose that C is a residue system of $F/\mathbb{Q}$ with $S(C)$ consisting of ideals whose norms are in $(N, KN]$ with multiplicity at most s, where $s \leqslant \exp(b\sqrt{\log N \log\log N})$ and $K = L(N, s)^{(1/2-\varepsilon)/c_F s}$. Then*

$$\delta(C) \geqslant \left(1 + O_{\varepsilon,b}\left(\frac{1}{(\log N)^{\lambda}}\right)\right)\alpha(C),$$

where λ is a positive constant depending only on ε and b.

We remark that we obtain our main theorems under the same conditions on b, c and ε as in Theorems 2–4 of [7]. Additional theorems which are analogues of those from [7] will be given later in Sections 4 and 5. In Section 4, we construct an exact covering system in a number field with the multiplicity of each modulus $\leqslant \exp(\sqrt{\log N \log\log N})$ and we also show that the density $\delta(C)$ can be considerably smaller than that of Theorem 3 provided K is sufficiently large. In Section 5, we study normal behaviors of $\delta(C)$ over random residue systems C with fixed $S(C)$.

2. Preliminary lemmas

In this section, we present lemmas which are analogues of all the lemmas in [7]. Throughout this paper, we have n a positive integer, p represents a prime. We use the Vinogradov notation $A \ll B$ which is the same as $A = O(B)$, and constants implied by the notation $\ll$, as with the notation O, may depend on the field F.

Let $F/\mathbb{Q}$ be a number field of degree d and let $\mathcal{O}_F$ be the ring of integers of F. Let C be a finite set of ordered pairs (I, r), which is a set of residue classes $r \pmod{I}$, where I is an ideal of $\mathcal{O}_F$ and $r \in \mathcal{O}_F$. We say such a set is a *residue system of* F. Let $S = S(C)$ denote the multiset of the moduli I appearing in C, and we call the number of times an ideal I appears in S the *multiplicity* of I. By $R(C)$ we denote the set of elements of $\mathcal{O}_F$ *not* congruent to $r \pmod{I}$ for any $(I, r) \in C$, and we denote the asymptotic density of $R(C)$ by $\delta(C)$. For $C = \{(I_1, r_1), \dots, (I_l, r_l)\}$, we let

$$\alpha(C) = \prod_{I \in S(C)} \left(1 - \frac{1}{\|I\|}\right) = \prod_{j=1}^{l} \left(1 - \frac{1}{\|I_j\|}\right), \qquad \beta(C) = \sum_{\substack{i<j \\ \|\gcd(I_i, I_j)\| > 1}} \frac{1}{\|I_i\| \|I_j\|},$$

where $\|I\|$ is the norm of the ideal I. We also let $f(n)$ and $g(n)$ denote the number of ideals of norm n and the number of prime ideals of norm n, respectively, as in the Introduction.

Lemma 2.1. *For an arbitrary residue system C of a number field $F/\mathbb{Q}$, we have $\delta(C) \geqslant \alpha(C) - \beta(C)$.*

Proof. Let $\alpha = \alpha(C)$ and $\beta = \beta(C)$. We denote $C' = \{(I_1, r_1), \dots, (I_{l-1}, r_{l-1})\}$, $C'' = \{(I_j, r_j)\colon\ j < l, \|\gcd(I_j, I_l)\| = 1\}$,

$$\alpha' = \alpha(C') = \prod_{j=1}^{l-1} \left(1 - \frac{1}{\|I_j\|}\right) \quad \text{and} \quad \beta' = \beta(C') = \sum_{\substack{i<j \leqslant l-1 \\ \|\gcd(I_i, I_j)\| > 1}} \frac{1}{\|I_i\| \|I_j\|}.$$

Now, follow the proof of Lemma 2.1 of [7] with the replacement of (n_i, r_i), $1/n_i$ and $\gcd(n_j, n_l)$ by (I_i, r_i), $1/\|I_i\|$ and $\|\gcd(I_j, I_l)\|$, respectively. $\square$

We can factor each modulus I as $I_{\underline{Q}} I_{\overline{Q}}$, where $I_{\underline{Q}}$ is the smallest ideal dividing I composed solely of prime ideals that lie over prime numbers in $[1, Q]$ with $Q \geqslant 1$, and $I_{\overline{Q}} = I / I_{\underline{Q}}$.

Lemma 2.2. *Let C be a residue system of a number field $F/\mathbb{Q}$. Let $Q \geqslant 2$ be arbitrary, and set*

$$M = \operatorname{lcm}\bigl\{I_{\underline{Q}}\colon\ I \in S(C)\bigr\}.$$

Let $\{(M, h_i)\colon\ 1 \leqslant i \leqslant \|M\|\}$ be a covering system of $\mathcal{O}_F$. For each h_i, let C_{h_i} be the set

$$C_{h_i} = \bigl\{(I_{\overline{Q}}, r)\colon\ (I, r) \in C,\ r \equiv h_i \pmod{I_{\underline{Q}}}\bigr\}.$$

Then

$$\delta(C) = \frac{1}{\|M\|} \sum_{i=1}^{\|M\|} \delta(C_{h_i}).$$

Proof. Using the Chinese Remainder Theorem, we can follow the same argument as in the proof of Lemma 3.1 of [7] (replacing $M, n_{\underline{Q}}$ and $n_{\overline{Q}}$ by $\|M\|, I_{\underline{Q}}$ and $I_{\overline{Q}}$, respectively). □

Now, we use the fact that $\|I_{\overline{Q}}\|$ has no prime factors $\leqslant Q$ to get an upper bound for the sum of $\beta(C_{h_i})$ as in the proof of Lemma 3.2 of [7].

Lemma 2.3. *Let $K > 1$, and suppose C is a residue system of a number field $F/\mathbb{Q}$ of degree d with $S(C)$ consisting of ideals whose norms are in the interval $(N, KN]$, each with multiplicity at most s. Let $Q \geqslant 2$, and define M and C_{h_i} as in Lemma 2.2. Then*

$$\frac{1}{\|M\|}\sum_{i=1}^{\|M\|}\beta(C_{h_i}) \ll \frac{s^2(1+\log K)^2\log^2 Q}{Q}. \tag{2.1}$$

Proof. For $J|M$, let S_J be the set of distinct ideals $I_{\overline{Q}} = I/\gcd(I, M)$, where $I \in S(C)$ and $I_{\underline{Q}} = \gcd(I, M) = J$. For $J, J' \mid M$, let

$$G(r, J, r', J') = \#\bigl\{1 \leqslant i \leqslant \|M\|\colon\ h_i \equiv r \pmod J,\ h_i \equiv r' \pmod{J'}\bigr\}.$$

Then

$$\frac{1}{\|M\|}\sum_{i=1}^{\|M\|}\beta(C_{h_i}) \leqslant \frac{1}{\|M\|}\sum_{\substack{J|M\\ J'|M}}\ \sum_{\substack{I\in S_J\\ I'\in S_{J'}\\ \|\gcd(I,I')\|>1}} \frac{1}{\|I\|\|I'\|}\sum_{\substack{(IJ,r)\in C\\ (I'J',r')\in C}} G(r, J, r', J').$$

We can see that $G(r, J, r', J')$ is either 0 or $\|M\|/\|\mathrm{lcm}[J, J']\|$, so the inner sum is at most

$$s^2\frac{\|M\|}{\|\mathrm{lcm}[J, J']\|}.$$

Next, let $\mathcal{P}$ denote a prime ideal, and let $P(n)$ and $P^-(n)$ denote the largest prime factor and the least prime factor of $n \geqslant 1$, respectively. Then

$$\begin{aligned}\sum_{\substack{I\in S_J\\ I'\in S_{J'}\\ \|\gcd(I,I')\|>1}} \frac{1}{\|I\|\|I'\|} &\leqslant \sum_{P(\|\mathcal{P}\|)>Q}\ \sum_{\substack{I\in S_J\\ I'\in S_{J'}\\ \mathcal{P}|I,\ \mathcal{P}|I'}} \frac{1}{\|I\|\|I'\|}\\ &= \sum_{P(\|\mathcal{P}\|)>Q}\frac{1}{\|\mathcal{P}\|^2}\Biggl(\sum_{\substack{N/\|\mathcal{P}J\|<\|I\|\leqslant KN/\|\mathcal{P}J\|\\ P^-(\|I\|)>Q}}\frac{1}{\|I\|}\Biggr)\Biggl(\sum_{\substack{N/\|\mathcal{P}J'\|<\|I'\|\leqslant KN/\|\mathcal{P}J'\|\\ P^-(\|I'\|)>Q}}\frac{1}{\|I'\|}\Biggr).\end{aligned}$$

Using Proposition 1 and partial summation, we obtain

$$\begin{aligned}\sum_{y<n\leqslant x}\frac{f(n)}{n} &= \frac{1}{x}\sum_{n\leqslant x} f(n) - \frac{1}{y}\sum_{n\leqslant y} f(n) + \int_y^x \frac{1}{t^2}\sum_{n\leqslant t} f(n)\,dt\\ &= c_F\log\frac{x}{y} + O\bigl(y^{-\frac{1}{d}}\bigr) \ll \log\frac{x}{y} + 1.\end{aligned} \tag{2.2}$$

Thus,

$$\sum_{\substack{N/\|\mathcal{P}J\|<\|I\|\leqslant KN/\|\mathcal{P}J\| \\ P^-(\|I\|)>Q}} \frac{1}{\|I\|} \leqslant \sum_{N/\|\mathcal{P}J\|<n\leqslant KN/\|\mathcal{P}J\|} \frac{f(n)}{n} \ll \log K + 1$$

and similarly with J', I' replacing J, I. We have the estimate

$$\sum_{P(\|\mathcal{P}\|)>Q} \frac{1}{\|\mathcal{P}\|^2} = \sum_{\substack{n\geqslant 1 \\ p>Q}} \frac{f(p^n)}{p^{2n}} \leqslant d \sum_{p>Q} \left(\frac{1}{p^2} + \frac{1}{p^4} + \cdots\right) \ll \frac{d}{Q \log Q},$$

which follows from the prime number theorem and partial summation.

Thus,

$$\sum_{\substack{I\in S_J \\ I'\in S_{J'} \\ \|\gcd(I,I')\|>1}} \frac{1}{\|I\|\|I'\|} \ll \frac{(1+\log K)^2}{Q \log Q},$$

so that

$$\frac{1}{\|M\|} \sum_{i=1}^{\|M\|} \beta(C_{h_i}) \ll \frac{s^2(1+\log K)^2}{Q \log Q} \sum_{\substack{J|M \\ J'|M}} \frac{1}{\|\mathrm{lcm}[J, J']\|} = \frac{s^2(1+\log K)^2}{Q \log Q} \sum_{u|M} \sum_{\mathrm{lcm}[J,J']=u} \frac{1}{\|u\|}.$$

Let $\tau(I)$ denote the number of divisors of an ideal I. Then

$$\begin{aligned}
\sum_{u|M} \sum_{\mathrm{lcm}[J,J']=u} \frac{1}{\|u\|} &= \sum_{u|M} \frac{\tau(u^2)}{\|u\|} \leqslant \prod_{P(\|\mathcal{P}\|)\leqslant Q} \left(1 + \frac{3}{\|\mathcal{P}\|} + \frac{5}{\|\mathcal{P}\|^2} + \cdots\right) \\
&= \prod_{n=1}^{d} \prod_{p\leqslant Q} \left(1 + \frac{3}{p^n} + \frac{5}{p^{2n}} + \cdots\right)^{g(p^n)} \\
&= \prod_{p\leqslant Q} \left(1 + \frac{3}{p} + \frac{5}{p^2} + \cdots\right)^{g(p)} \prod_{n=2}^{d} \prod_{p\leqslant Q} \left(1 + \frac{3}{p^n} + \frac{5}{p^{2n}} + \cdots\right)^{g(p^n)} \\
&\ll \prod_{p\leqslant Q} \left(1 + \frac{3}{p} + \frac{5}{p^2} + \cdots\right)^{g(p)} \\
&\leqslant \exp\left(\sum_{p\leqslant Q} g(p)\left(\frac{3}{p} + \frac{5}{p^2} + \cdots\right)\right) \ll \exp\left(3 \sum_{p\leqslant Q} \frac{g(p)}{p}\right) \\
&\leqslant \exp\left(3 \sum_{n\leqslant Q} \frac{g(n)}{n}\right) \ll \log^3 Q,
\end{aligned}$$

since $\sum_{\|\mathcal{P}\|\leqslant Q} 1/\|\mathcal{P}\| = \sum_{n\leqslant Q} g(n)/n = \log\log Q + O(1)$ by Proposition 2 and partial summation. This completes the proof. $\square$

We can also obtain a lower bound for the sum of $\alpha(C_{h_i})$ using those I's in $S(C)$ such that $P(\|I\|) \leqslant Q$.

Lemma 2.4. *Let C be an arbitrary residue system of $F/\mathbb{Q}$. For $Q \geqslant 2$, define M and C_{h_i} as in Lemma 2.2. Let $C' = \{(I,r) \in C\colon\ I|M\} = \{(I,r) \in C\colon\ P(\|I\|) \leqslant Q\}$ and suppose $\delta(C') > 0$. Then*

$$\frac{1}{M}\sum_{i=1}^{\|M\|} \alpha(C_{h_i}) \geqslant \bigl(\alpha(C)\bigr)^{(1+1/Q)/\delta(C')}.$$

Proof. Note that $\mathcal{O}_L \in S(C_{h_i})$ if and only if there is a pair $(I,r) \in C'$ with $h_i \equiv r \pmod I$. Let

$$\mathcal{M}' = \bigl\{1 \leqslant i \leqslant \|M\|\colon\ \mathcal{O}_F \notin S(C_{h_i})\bigr\}, \quad M' = |\mathcal{M}'|.$$

Then

$$\frac{M'}{\|M\|} = \delta(C'). \tag{2.3}$$

Now, follow the same argument as in the proof of Lemma 3.3 of [7] replacing $M,\ 1/n,\ 1/n',\ 1/n_{\underline{Q}}$ and $1/n_{\overline{Q}}$ by $\|M\|,\ 1/\|I\|,\ 1/\|I'\|,\ 1/\|I_{\underline{Q}}\|$ and $1/\|I_{\overline{Q}}\|$, respectively. □

Now, combining the above two lemmas yields the following.

Lemma 2.5. *Suppose $K > 1$, N is a positive integer, and C is a residue system of $F/\mathbb{Q}$ with $S(C)$ consisting of ideals whose norms are in $(N, KN]$, each with multiplicity at most s. Let $Q \geqslant 2$, and as in Lemma 2.4, let $C' = \{(I,r) \in C\colon\ P(\|I\|) \leqslant Q\}$. If $\delta(C') > 0$, then*

$$\delta(C) \geqslant \alpha(C)^{(1+1/Q)/\delta(C')} + O\left(\frac{s^2(1+\log K)^2\log^2 Q}{Q}\right),$$

where the implied constant depends on F only.

Proof. Using the same definition of M and C_{h_i} as in Lemma 2.2 and by Lemmas 2.1–2.4, we have

$$\begin{aligned}\delta(C) = \frac{1}{M}\sum_{i=1}^{\|M\|}\delta(C_{h_i}) &\geqslant \frac{1}{\|M\|}\sum_{i=1}^{\|M\|}\alpha(C_{h_i}) - \frac{1}{\|M\|}\sum_{i=1}^{\|M\|}\beta(C_{h_i}) \\ &\geqslant \alpha(C)^{(1+1/Q)/\delta(C')} + O\left(\frac{s^2(1+\log K)^2\log^2 Q}{Q}\right).\end{aligned}$$

Thus, we have the lemma. □

Next, we show an analogue of Lemma 4.1 of [7] which is about smooth numbers.

Our result is more complicated to prove because we need to understand $f(n)$ at smooth arguments n.

Lemma 2.6. *Let $F/\mathbb{Q}$ be a number field of degree d with the ring of integers $\mathcal{O}_F$. Suppose $Q \geqslant 2$ and $Q < N \leqslant \exp(\exp(\log^{2/5} Q))$. Let $f(n)$ be the number of ideals of norm n in $\mathcal{O}_F$, then*

$$\sum_{\substack{n>N\\ P(n)\leqslant Q}} \frac{f(n)}{n} \ll (\log Q)e^{-u\log u}, \quad \textit{where } u = \frac{\log N}{\log Q}.$$

Proof. We use Corollary 2.3 of [21] with $\kappa = 1$, and $L_{1/5}(z) = \exp\{(\log z)^{2/5}\}$.
Using Proposition 2,

$$\sum_{p \leqslant z} f(p)\log p = \sum_{\|\mathcal{P}\| \leqslant z} \log\|\mathcal{P}\| - \sum_{\substack{\|\mathcal{P}\| \leqslant z \\ \|\mathcal{P}\| = q^l \\ l \geqslant 2}} \log\|\mathcal{P}\|$$

$$= \sum_{\|\mathcal{P}\| \leqslant z} \log\|\mathcal{P}\| + O(\sqrt{z}) = z + O\left(\frac{z}{\exp(\log z)^{1/13}}\right).$$

Thus, for some constant C, if $z > 1$, then

$$\left|\sum_{p \leqslant z} f(p)\log p - z\right| \leqslant Cz/L_{1/5}(z),$$

which is (2.1) of [21]. Since (1.8) of [21] also holds for some $A > 0$ and $\eta \in (0, 1/2)$, $f \in \mathscr{M}_1(A, C, \eta, L_{1/5})$. By Corollary 2.3 of [21],

$$\sum_{\substack{n \leqslant t \\ P(n) \leqslant Q}} f(n) \ll \frac{t}{u_t^{u_t}},$$

where $u_t = \log t / \log Q$, provided $Q \leqslant t \leqslant t_0 = Q^{L_{1/5}(Q)}$, since $\rho_1(u) = \rho(u) \ll u^{-u}$ [9, Corollary 2.3].
Let $Q'(t) = \exp\{(\log\log t)^{5/2}\}$. Note that if $t > t_0$, then $Q'(t) \geqslant Q$. Thus, for $t > t_0$,

$$\sum_{\substack{n \leqslant t \\ P(n) \leqslant Q}} f(n) \leqslant \sum_{\substack{n \leqslant t \\ P(n) \leqslant Q'(t)}} f(n) \ll t/u'^{u'}, \quad \text{where } u' = u'(t) = \frac{\log t}{\log Q'(t)},$$

since $1 \leqslant u'(t) \leqslant L_{1/5}(Q')$. Let i_0 be the largest integer such that $NQ^{i_0} \leqslant t_0$. Then,

$$\begin{aligned}
\sum_{\substack{n > N \\ P(n) \leqslant Q}} \frac{f(n)}{n} &= \int_N^\infty \frac{1}{t^2} \sum_{\substack{N < n \leqslant t \\ P(n) \leqslant Q}} f(n)\,dt \\
&\leqslant \sum_{i=0}^{i_0 - 1} \int_{NQ^i}^{NQ^{i+1}} \frac{1}{t^2} \sum_{\substack{n \leqslant t \\ P(n) \leqslant Q}} f(n)\,dt + \int_{NQ^{i_0}}^{t_0} \frac{1}{t^2} \sum_{\substack{n \leqslant t \\ P(n) \leqslant Q}} f(n)\,dt + \int_{t_0}^\infty \frac{1}{t^2} \sum_{\substack{n \leqslant t \\ P(n) \leqslant Q}} f(n)\,dt \\
&\ll \sum_{i \geqslant 0} \frac{\log Q}{(u+i)^{u+i}} + \int_{NQ^{i_0}}^{t_0} \frac{1}{t u_t^{u_t}}\,dt + \int_{t_0}^\infty \frac{1}{t\log^2 t} \cdot \frac{\log^2 t}{u'^{u'}}\,dt \\
&\ll \frac{\log Q}{u^u} + \int_{NQ^{i_0}}^{NQ^{i_0+1}} \frac{1}{t u_t^{u_t}}\,dt + \frac{\log t_0}{u'(t_o)^{u'(t_0)}} \ll \frac{\log Q}{u^u},
\end{aligned}$$

implying the lemma. □

Lastly, we present a lemma which will be needed in the proof of Theorem 2 in Section 3.

Lemma 2.7. *Suppose s is a positive integer and C is a residue system of a number field $F/\mathbb{Q}$ with $S(C)$ consisting of ideals whose norms are in $(1, B]$ with multiplicity at most s. Let*

$$C_0 = \left\{(I, r) \in C\colon\ \mathcal{P}|I \Rightarrow \|\mathcal{P}\| \leqslant \sqrt{s\nu B}\right\},$$

where ν is a constant depending on F such that $\sum_{n \leqslant x} f(n) \leqslant \nu x$ for all x. (Note that Proposition 1 guarantees that such ν exists, and $\nu \geqslant c_F$.) If $\delta(C_0) > 0$, then $\delta(C) > 0$.

Proof. Suppose $\delta(C_0) > 0$. Let P be the set of prime ideals whose norms are in $(\sqrt{s\nu B}, B]$ and let l be the least common multiple of all $I \in S(C_0)$. Let $\mathcal{P} \in P$. Since the number of ideals $I \in S(C)$ such that $\mathcal{P}|I$ is $\leqslant s \sum_{n \leqslant B/\|\mathcal{P}\|} f(n) \leqslant s\nu B/\|\mathcal{P}\| < \|\mathcal{P}\|$, there are at most $\|\mathcal{P}\| - 1$ ideals I such that $\mathcal{P}|I$. Call them $I_1, \ldots, I_t$, and let $r_1, \ldots, r_t$ be the corresponding residue classes. Then there is a choice for $b = b(\mathcal{P})$ such that if $x \equiv b \pmod{\mathcal{P}}$, then x is not covered by any of the congruences $x \equiv r_j \pmod{I_j}$ with $1 \leqslant j \leqslant t$.

By assumption, there is a residue class a mod l in $R(C_0)$. Let A be a solution to the system $A \equiv a \pmod{l}$ and $A \equiv b(\mathcal{P}) \pmod{\mathcal{P}}$ for each prime ideal $\mathcal{P} \in P$. Such A exists via the Chinese Remainder Theorem. Then we have $A \not\equiv r \pmod{I}$ for each $(I, r) \in C_0$. Furthermore, for each prime $\mathcal{P} \in P$ and $(I, r) \in C$ with $\mathcal{P}|I$, $A \not\equiv r \pmod{I}$. Since this exhausts the pairs $(I, r) \in C$, we have $A \in R(C)$, and it completes the proof. $\square$

3. Proof of Theorems 1, 2 and 3

Proof of Theorem 1. We can repeat the same proof as that of Theorem 2 of [7] using Lemmas 2.6 and 2.5 (instead of Lemmas 4.1 and 3.4 of [7]). $\square$

Proof of Theorem 2. We can suppose that $\varepsilon > 0$ is sufficiently small and $K \geqslant 2$. Let

$$C_0 = \left\{(I, r) \in C\colon\ \mathcal{P}|I \Rightarrow \|\mathcal{P}\| \leqslant \sqrt{s\nu KN}\right\},$$

with ν as in Lemma 2.7. Then, by (2.2),

$$\begin{aligned}
\sum_{I \in S(C_0)} \frac{1}{\|I\|} &\leqslant s \sum_{N < \|I\| \leqslant KN} \frac{1}{\|I\|} - s \sum_{\substack{N < \|I\| \leqslant KN \\ \exists \mathcal{P}|I\colon\ \|\mathcal{P}\| > \sqrt{s\nu B}}} \frac{1}{\|I\|} \\
&= s \sum_{N < n \leqslant KN} \frac{f(n)}{n} - s \sum_{\sqrt{s\nu KN} < \|\mathcal{P}\| \leqslant KN} \frac{1}{\|\mathcal{P}\|} \sum_{N/\|\mathcal{P}\| < \|I'\| \leqslant KN/\|\mathcal{P}\|} \frac{1}{\|I'\|} \\
&= sc_F \log K + O\left(\frac{s}{N^{1/d}}\right) - s \sum_{\sqrt{s\nu KN} < \|\mathcal{P}\| \leqslant KN} \frac{1}{\|\mathcal{P}\|} \sum_{N/\|\mathcal{P}\| < n \leqslant KN/\|\mathcal{P}\|} \frac{f(n)}{n}.
\end{aligned}$$

Now,

$$\sum_{N/\|\mathcal{P}\| < n \leqslant KN/\|\mathcal{P}\|} \frac{f(n)}{n} = \begin{cases} c_F \log K + O((\|\mathcal{P}\|/N)^{\frac{1}{d}}), & \|\mathcal{P}\| \leqslant N, \\ c_F \log(KN/\|\mathcal{P}\|) + O(1), & N < \|\mathcal{P}\| \leqslant KN. \end{cases}$$

Thus,

$$\sum_{\sqrt{s\nu KN} < \|\mathcal{P}\| \leqslant KN} \frac{1}{\|\mathcal{P}\|} \sum_{N/\|\mathcal{P}\| < n \leqslant KN/\|\mathcal{P}\|} \frac{f(n)}{n}$$

$$
\begin{aligned}
&= \sum_{\sqrt{s\nu KN}<\|\mathcal{P}\|\leqslant N}\left(\frac{c_F\log K}{\|\mathcal{P}\|}+O\left(\frac{1}{N^{\frac{1}{d}}\|\mathcal{P}\|^{1-\frac{1}{d}}}\right)\right)+c_F\sum_{N<\|\mathcal{P}\|\leqslant KN}\frac{\log K}{\|\mathcal{P}\|}\\
&\quad+c_F\sum_{N<\|\mathcal{P}\|\leqslant KN}\frac{\log N-\log\|\mathcal{P}\|+O(1)}{\|\mathcal{P}\|}\\
&= c_F\log K\sum_{\sqrt{s\nu KN}<\|\mathcal{P}\|\leqslant KN}\frac{1}{\|\mathcal{P}\|}+c_F\sum_{N<\|\mathcal{P}\|\leqslant KN}\frac{\log N}{\|\mathcal{P}\|}-c_F\sum_{N<\|\mathcal{P}\|\leqslant KN}\frac{\log\|\mathcal{P}\|}{\|\mathcal{P}\|}\\
&\quad+O\left(\sum_{\sqrt{s\nu KN}<\|\mathcal{P}\|\leqslant N}\frac{1}{N^{\frac{1}{d}}\|\mathcal{P}\|^{1-\frac{1}{d}}}\right)+O\left(\sum_{N<\|\mathcal{P}\|\leqslant KN}\frac{1}{\|\mathcal{P}\|}\right).
\end{aligned}
$$

By Proposition 2 and partial summation,

$$
\begin{aligned}
\sum_{y<\|\mathcal{P}\|\leqslant x}\frac{1}{\|\mathcal{P}\|}&=\sum_{y<n\leqslant x}\frac{g(n)}{n}=\log\log x-\log\log y+O\left(\frac{1}{\log y}\right),\\
\sum_{y<\|\mathcal{P}\|\leqslant x}\frac{\log\|\mathcal{P}\|}{\|\mathcal{P}\|}&=\sum_{y<n\leqslant x}\frac{g(n)\log n}{n}=\log\frac{x}{y}+O\left(\frac{1}{\log y}\right),\\
\sum_{y<\|\mathcal{P}\|\leqslant x}\frac{1}{\|\mathcal{P}\|^{1-1/d}}&=\sum_{y<n\leqslant x}\frac{g(n)}{n^{1-1/d}}=O\left(\frac{x^{\frac{1}{d}}}{\log y}\right).
\end{aligned}
$$

So,

$$
\begin{aligned}
&\sum_{\sqrt{s\nu KN}<\|\mathcal{P}\|\leqslant KN}\frac{1}{\|\mathcal{P}\|}\sum_{N/\|\mathcal{P}\|<n\leqslant KN/\|\mathcal{P}\|}\frac{f(n)}{n}\\
&= c_F\log K(\log\log KN-\log\log\sqrt{s\nu KN})+c_F\log N(\log\log KN-\log\log N)\\
&\quad- c_F\log K+O(\log\log KN-\log\log N)+O(1)\\
&= c_F\log K\left(\log 2-\log\left(1+\frac{\log s\nu}{\log KN}\right)\right)+c_F\log N\log\left(1+\frac{\log K}{\log N}\right)-c_F\log K\\
&\quad+O\left(\log\left(1+\frac{\log K}{\log N}\right)\right)+O(1)\\
&= c_F\log 2\log K+O\left(\frac{\log K\log s\nu}{\log KN}\right)+O(1)=\big(c_F\log 2+o(1)\big)\log K.
\end{aligned}
$$

Thus,

$$
\sum_{I\in S(C_0)}\frac{1}{\|I\|}\leqslant sc_F\big((1-\log 2)+o(1)\big)\log K.
$$

Since

$$
-\log\alpha(C_0)\leqslant\sum_{I\in S(C_0)}\frac{1}{\|I\|}+O\left(s\sum_{n>N}\frac{f(n)}{n^2}\right)=\sum_{I\in S(C_0)}\frac{1}{\|I\|}+O\left(\frac{s}{N}\right),
$$

we have

$$-\log\alpha(C_0) \leqslant sc_F\big(1-\log 2+o(1)\big)\log K \leqslant \big(1-(1-\log 2)\varepsilon+o(1)\big)\log L(N,s).$$

Let $Q = L(N,s)^{1-\lambda}$, where $\lambda = \frac{1}{4}((1-\log 2)\varepsilon - 4b^2)$, and let $C' = \{(n,r) \in C_0\colon\ P(n) \leqslant Q\}$. Using Lemma 2.6 yields

$$\delta(C') = 1 + O\Bigg(s \sum_{\substack{n>N\\ P(n)\leqslant Q}} \frac{f(n)}{n}\Bigg) = 1+o(1) \quad (N\to\infty).$$

Thus,

$$\alpha(C_0)^{(1+1/Q)/\delta(C')} \gg L(N,s)^{-1+(1-\log 2)\varepsilon-\lambda}.$$

On the other hand,

$$\frac{s^2(1+\log K)^2\log^2 Q}{Q} \ll L(N,s)^{-1+\lambda}s^2\big(\log L(N,s)\big)^4 \ll L(N,s)^{-1+4b^2+2\lambda}.$$

By Lemma 2.5, we have $\delta(C_0) > 0$ for N sufficiently large. Hence $\delta(C) > 0$ by Lemma 2.7. □

Proof of Theorem 3. By (2.2), we have

$$-\log\alpha(C) \leqslant s \sum_{N<n\leqslant KN} \left(\frac{f(n)}{n} + \frac{f(n)}{n^2}\right) \leqslant s\left(c_F \log K + O\left(\frac{1}{N^{1/d}}\right)\right).$$

So,

$$\alpha(C) \gg K^{-sc_F} = L(N,s)^{-1/2+\varepsilon}.$$

Let $\lambda = \frac{1}{3}(\varepsilon - 4b^2)$ and $Q = L(N,s)^{1/2-\lambda}$. Let $u = \log N/\log Q$, and let C' be as in Lemma 2.5. By Lemma 2.6, we have

$$1-\delta(C') \ll \frac{s\log Q}{u^u} \ll \frac{s\log N}{(s\log N)^{2+\lambda}} = \frac{1}{(s\log N)^{1+\lambda}},$$

so that $1/\delta(C') = 1 + O((s\log N)^{-1-\lambda})$. Using $|\log\alpha(C)| \leqslant \log N$, we have

$$\alpha(C)^{(1+1/Q)/\delta(C')} = \left(1 + O\left(\frac{1}{(\log N)^\lambda}\right)\right)\alpha(C).$$

By Lemma 2.5 it suffices to show that

$$\frac{s^2(1+\log K)^2\log^2 Q}{Q} = O\left(\frac{\alpha(C)}{(\log N)^\lambda}\right).$$

But, for large N we have $s^2\log^4 L(N,s) \leqslant L(N,s)^{4b^2+\lambda}$. Thus,

$$\begin{aligned}\frac{s^2(1+\log K)^2\log^2 Q}{Q} &\ll \frac{s^2\log^4 L(N,s)}{L(N,s)^{1/2-\lambda}} \ll \frac{1}{L(N,s)^{1/2-2\lambda-4b^2}}\\ &\ll \frac{1}{L(N,s)^{1/2-\varepsilon+\lambda}} \ll \frac{\alpha(C)}{L(N,s)^\lambda}. \qquad \square\end{aligned}$$

4. Exact coverings and near coverings in number fields

In this section, we prove analogues of Theorems 5 and 6 of [7]. As in [7], they imply that there is an exact covering system of an arbitrary number field, where each modulus I has norm $\geqslant N$ and multiplicity near the upper bound given in Theorems 1–3, and the density $\delta(C)$ can be considerably smaller than that given in Theorem 3 if we allow K to be sufficiently large.

Theorem 4. *Let $F/\mathbb{Q}$ be a number field with the ring of integers $\mathcal{O}_F$. For sufficiently large N and $s = \exp(\sqrt{\log N \log\log N})$, there exists an exact covering system of F with squarefree moduli whose norm is greater than N such that the multiplicity of each modulus does not exceed s.*

Proof. We follow the key idea from the proof of Theorem 5 of [7] to construct the desired covering system, and we also use the method from an older preprint version of [7] based on the Remark 4 of [7] to complete the proof.

Let $\mathcal{P}$ denote a prime ideal of $\mathcal{O}_F$, and define a sequence $\{X_j\}$ by

$$X_0 = 1 \quad \text{and} \quad X_{j+1} = \min\left\{x\colon \sum_{X_j < \|\mathcal{P}\| \leqslant x} \left[\frac{x}{\|\mathcal{P}\|}\right] \geqslant X_j\right\} \quad \text{with } j \geqslant 0,$$

where $[x]$ denotes the greatest integer which is $\leqslant x$. Let

$$P_j = \{\mathcal{P}\colon\ X_{j-1} < \|\mathcal{P}\| \leqslant X_j\}.$$

First, for $J \geqslant 1$ and $s = X_J$, we construct an exact covering system C_J with squarefree moduli of the form $\mathcal{P}_1 \cdots \mathcal{P}_J$ with $\mathcal{P}_i \in P_i$ with the multiplicity of each modulus $\leqslant s$. Note that such moduli have norms greater than

$$N_J = \prod_{j=0}^{J-1} X_j.$$

We construct C_J through induction on J. Choose any prime ideal $\mathcal{P}$ in P_1. Then, we can find r_i's from $\mathcal{O}_F$ such that $C_1 = \{(\mathcal{P}, r_1), \ldots, (\mathcal{P}, r_{\|\mathcal{P}\|})\}$ is an exact covering system of F. We can see that C_1 satisfies the above conditions with $J = 1$.

Now, suppose that we have C_J as above for some $J \leqslant 1$. Fix a modulus $I = \mathcal{P}_1 \cdots \mathcal{P}_J$, and let $(I, r_1), \ldots, (I, r_t)$ be all the pairs in C_J corresponding to I. Note that $t \leqslant X_J$. Let $P_{J+1} = \{\mathcal{Q}_1, \ldots, \mathcal{Q}_m\}$. Replace each (I, r_i), $i \leqslant [X_{J+1}/\|\mathcal{Q}_1\|]$, by the $\|\mathcal{Q}_1\|$ pairs $(I\mathcal{Q}_1, r_i + a_k)$, where $I = \bigcup_{k=1}^{\|\mathcal{Q}_1\|}(a_k + I\mathcal{Q}_1)$. Note that the multiplicity of the modulus $I\mathcal{Q}_1$ is $[X_{J+1}/\|\mathcal{Q}_1\|]\|\mathcal{Q}_1\| \leqslant X_{J+1}$ and $r_i + I = \bigcup_{k=1}^{\|\mathcal{Q}_1\|}(r_i + a_k + I\mathcal{Q}_1)$.

Next, replace each (I, r_i), $[X_{J+1}/\|\mathcal{Q}_1\|] < i \leqslant [X_{J+1}/\|\mathcal{Q}_1\|] + [X_{J+1}/\|\mathcal{Q}_2\|]$, with the $\|\mathcal{Q}_2\|$ pairs $(I\mathcal{Q}_2, r_i + b_k)$, where $I = \bigcup_{k=1}^{\|\mathcal{Q}_2\|}(b_k + I\mathcal{Q}_2)$. Similarly, the multiplicity of the modulus $I\mathcal{Q}_2$ is $\leqslant X_{J+1}$ and $r_i + I = \bigcup_{k=1}^{\|\mathcal{Q}_2\|}(r_i + b_k + I\mathcal{Q}_2)$. Continuing this construction, all the pairs $(I, r_1), \ldots, (I, r_t)$ can be replaced with sets of residue classes with moduli of the form $I\mathcal{Q}_i$, since

$$t \leqslant X_J \leqslant \sum_{X_J < \|\mathcal{P}\| \leqslant X_{J+1}} [X_{J+1}/\|\mathcal{P}\|].$$

Applying this procedure for each $I \in S(C_J)$ completes the inductive construction of C_{J+1}.

In order to complete the proof, it suffices to show that for sufficiently large N, we can take J such that

$$N \leqslant N_J \quad \text{and} \quad \log^2 s = \log^2 X_J \leqslant \log N \log\log N.$$

We begin by showing that if $\varepsilon \in (0,1)$ and j is sufficiently large, say $j \geqslant j(\varepsilon)$, then

$$X_{j+1} \leqslant X_j \frac{(1+\varepsilon)\log X_j}{\log\log X_j}. \tag{4.1}$$

Set

$$x = \left[X_j \frac{(1+\varepsilon)\log X_j}{\log\log X_j} \right].$$

By Proposition 2, we have

$$\begin{aligned}
\sum_{X_j < \|\mathcal{P}\| \leqslant x} \big[x/\|\mathcal{P}\|\big] &\geqslant x \sum_{X_j < n \leqslant x} \frac{g(n)}{n} - \sum_{X_j < n \leqslant x} g(n) \\
&= x\big(\log\log x - \log\log X_j + O(1/\log X_j)\big) + O(x/\log x) \\
&= x \log\left(1 + \frac{\log\log X_j + O(\log\log\log X_j)}{\log X_j}\right) + O(x/\log X_j) \\
&\geqslant x\left(\frac{(1-\varepsilon/3)(\log\log X_j)}{\log X_j}\right) + O(x/\log X_j) \\
&\geqslant x\left(\frac{(1-\varepsilon/2)(\log\log X_j)}{\log X_j}\right) > X_j.
\end{aligned}$$

This completes the proof of (4.1).

Next, we show that for every $\varepsilon \in (0,1)$, J sufficiently large (depending on ε), and $j \leqslant J$, we have

$$\log X_j \geqslant \log X_J - (J-j)(\log\log X_J - \log\log\log X_J + \varepsilon). \tag{4.2}$$

We consider first the case that $j \geqslant j(\varepsilon)$, where we choose $j(\varepsilon)$ such that $X_{j(\varepsilon)} \geqslant e^e$. It follows from (4.1) that in the case that $j \in [j(\varepsilon), J)$, we have

$$\begin{aligned}
\log X_{j+1} &\leqslant \log X_j + \log\log X_j - \log\log\log X_j + \log(1+\varepsilon) \\
&\leqslant \log X_j + \log\log X_J - \log\log\log X_J + \varepsilon.
\end{aligned}$$

Therefore, (4.2) holds for all $j \in [j(\varepsilon), J)$. On the other hand, if J is large, the left side of (4.2) decreases by a smaller amount as j changes from $j(\varepsilon)$ to 1 by comparison to the amount of decrease on the right side of (4.2). Hence, (4.2) in fact holds for all $j \leqslant J$ provided J is sufficiently large.

Now, we complete the proof of the theorem. Fix $\varepsilon \in (0,1)$. Let N be large, and take J so that

$$N_{J-1} = \prod_{j=1}^{J-2} X_j < N \leqslant N_J = \prod_{j=1}^{J-1} X_j.$$

Let $s = X_J$, and set

$$\Delta = \log\log X_J - \log\log\log X_J + \varepsilon = \log\log s - \log\log\log s + \varepsilon.$$

From (4.2), we have

$$\log N_{J-1} \geqslant \sum_{\substack{i\geqslant 2\\ i\Delta\leqslant\log s}} (\log s - i\Delta) \geqslant \frac{\log^2 s}{2\Delta+\varepsilon}.$$

Let Y denote the expression on the right above. Then

$$\log Y = 2\log\log s - \log\log\log s + O(1) > 2\Delta + \varepsilon.$$

Thus,

$$\log N \geqslant \log N_{J-1} \geqslant \frac{\log^2 s}{\log Y} \geqslant \frac{\log^2 s}{\log\log N},$$

and this completes the proof of the theorem. □

Let C be a residue system of a number field $F/\mathbb{Q}$, where $S(C)$ consists of distinct ideals whose norms are in $(N, KN]$. Then, using (2.2),

$$\begin{aligned}
\alpha(C) = \prod_{I\in S(C)}\left(1-\frac{1}{\|I\|}\right) &\geqslant \prod_{N<n\leqslant KN}\left(1-\frac{1}{n}\right)^{f(n)}\\
&\geqslant \exp\left(-\sum_{N<n\leqslant KN}\left(\frac{f(n)}{n}+\frac{f(n)}{n^2}\right)\right)\\
&= \exp\bigl(-c_F\log K + O\bigl(N^{-1/d}\bigr)\bigr).
\end{aligned}$$

Thus, if K is not too large, then Theorem 3 implies that $\delta(C)$ has a lower bound approximately $1/K^{c_F}$. The following theorem shows that, when we allow K to be much larger than N, C can be chosen so that $\delta(C)$ is considerably smaller than $1/K^{c_F}$.

Theorem 5. *Suppose N and K are integers sufficiently large depending on $F/\mathbb{Q}$. Then there is some residue system C consisting of distinct moduli whose norms are from $(N, KN]$ such that*

$$\delta(C) \leqslant \frac{1}{K^{c_F}}\exp\left(-c_F^2\frac{\log K}{3N}\right).$$

Before proving Theorem 5, we present a lemma about the expected value of $\delta(C)$. Let T be a set of ideals and let $\mathcal{C}(T)$ be the set of residue systems C with $S(C) = T$. Define

$$W_0(T) = \prod_{I\in T} I \quad\text{and}\quad W(T) = \#\mathcal{C}(T) = \prod_{I\in T}\|I\| = \bigl\|W_0(T)\bigr\|.$$

Lemma 4.1. *Let T be a set of distinct ideals. Then the expected value of $\delta(C)$ over $C\in\mathcal{C}(T)$, denoted $\mathbf{E}\delta(C)$, is $\prod_{I\in T}(1-1/\|I\|)$.*

Proof. Put $W = W(T)$ and $W_0 = W_0(T)$. Let $(W_0, m_1), \ldots, (W_0, m_{\|W_0\|})$ be an exact covering system. Since the number of systems $C\in\mathcal{C}(T)$ with $m_i\in R(C)$ is $\prod_{I\in T}(\|I\|-1)$, we have

$$\sum_{C\in\mathcal{C}(T)}\delta(C) = \sum_{C\in\mathcal{C}(T)}\frac{1}{W}\sum_{\substack{i=1\\ m_i\in R(C)}}^{W} 1 = \frac{1}{W}\sum_{i=1}^{W}\sum_{\substack{C\in\mathcal{C}(T)\\ m_i\in R(C)}} 1 = \frac{1}{W}\sum_{i=1}^{W}\prod_{I\in T}\bigl(\|I\|-1\bigr) = \prod_{I\in T}\bigl(\|I\|-1\bigr).$$

Dividing the equation above by W, we complete the proof. □

Proof of Theorem 5. We follow the construction of covering systems described in the proof of Theorem 6 of [7]: We will randomly choose the values of $r(I)$ for I with $N < \|I\| \leqslant 2N$ so that each residue class modulo I is taken with the same probability $1/\|I\|$ and the variables $r(I)$ are independent. We then select the remaining values of $r(I)$ for I with $2N < \|I\| \leqslant KN$ via a greedy algorithm. It suffices to show that, under our construction, the expected value of $\delta(C)$ over all randomly chosen values of $r(I)$ for I with $N < \|I\| \leqslant 2N$ is

$$\leqslant \frac{1}{K^{c_F}} \exp\left(-c_F^2 \frac{\log K}{3N}\right).$$

Let $C_{2N} = \{(I, r(I)):\ N < \|I\| \leqslant 2N\}$, where each $r(I)$ is selected randomly. Using Lemma 4.1 and (2.2), we have

$$\begin{aligned}\mathbf{E}\delta(C_{2N}) = \prod_{I\in T}\bigl(1 - 1/\|I\|\bigr) &\leqslant \exp\left(-\sum_{N<n\leqslant 2N} \frac{f(n)}{n}\right)\\ &= \exp\bigl(-c_F \log 2 + O\bigl(N^{-1/d}\bigr)\bigr).\end{aligned}$$

Thus, by the arithmetic mean–geometric mean inequality, it follows that

$$\mathbf{E}\log\delta(C_{2N}) \leqslant -c_F\log 2 + O\bigl(N^{-1/d}\bigr). \tag{4.3}$$

Now, we describe how to choose $r(J)$, where $2N < \|J\| \leqslant KN$. First, let $C_j = \{(I, r(I)):\ N < \|I\| \leqslant j\}$ and let $I_{j,1}, \dots, I_{j,f(j)}$ be the ideals whose norm is j. Here, if $f(j) = 0$, then $C_j = C_{j-1}$. Note that the residue class $r(J) \pmod J$ contains $r \pmod{I_{j,i}}$ when $J | I_{j,i}$ and $r \equiv r(J) \pmod J$. Thus, if $I_{j,i}$ has a divisor J with $N < \|J\| \leqslant 2N$, then there are residue classes modulo J not intersecting $R(C_{j-1})$. Let

$$D(j,i) = \bigl\{J:\ J|I_{j,i},\ N < \|J\| \leqslant 2N\bigr\}, \qquad \widetilde{C}_{j,i} = \bigl\{\bigl(J, r(J)\bigr):\ J \in D(j,i)\bigr\}.$$

Let $h(j,i)$ be the number of residue classes $r \pmod{I_{j,i}}$ for which $r \not\equiv r(J) \pmod J$ for each $J \in D(j,i)$. Note that if $h(j,i) = 0$ or 1 for some i, then we have $R(C_{j-1}) = \emptyset$ or $R(C_j) = \emptyset$. Thus, we assume that $h(j,i) > 1$ for all i. Then, we can select $r(J)$ from the $h(j,i)$ choices so that

$$\delta(C_j) \leqslant \prod_{i=1}^{f(j)}\left(1 - \frac{1}{h(j,i)}\right)\delta(C_{j-1}). \tag{4.4}$$

Using linearity of expectation, we obtain

$$\mathbf{E}\log\delta(C_j) - \mathbf{E}\log\delta(C_{j-1}) \leqslant \mathbf{E}\sum_{i=1}^{f(j)}\log\left(1 - \frac{1}{h(j,i)}\right) \leqslant -\sum_{i=1}^{f(j)}\mathbf{E}\left(\frac{1}{h(j,i)}\right). \tag{4.5}$$

Also, Lemma 4.1 implies

$$\mathbf{E}\delta(\widetilde{C}_{j,i}) = \prod_{J\in D(j,i)}\left(1 - \frac{1}{\|J\|}\right).$$

We can see that $\delta(\widetilde{C}_{j,i}) = h(j,i)/j$, since $I_{j,i}$ is a common multiple of the members of $D(j,i)$. Thus,

$$\mathbf{E}h(j,i) = j\mathbf{E}\delta(\widetilde{C}_{j,i}) = j\prod_{J\in D(j,i)}\left(1 - \frac{1}{\|J\|}\right),$$

and using the arithmetic mean–harmonic mean inequality, we also have

$$\mathbf{E}\left(\frac{1}{h(j,i)}\right) \geqslant j^{-1} \prod_{J \in D(j,i)} \left(1 - \frac{1}{\|J\|}\right)^{-1} \geqslant \frac{1}{j} + \sum_{J \in D(j,i)} \frac{1}{\|J\| j}.$$

From (4.5), we obtain

$$\mathbf{E} \log \delta(C_j) - \mathbf{E} \log \delta(C_{j-1}) \leqslant -\frac{f(j)}{j} - \sum_{i=1}^{f(j)} \sum_{J \in D(j,i)} \frac{1}{\|J\| j},$$

and thus, using (2.2),

$$\begin{aligned}
\mathbf{E} \log \delta(C) - \mathbf{E} \log \delta(C_{2N}) &\leqslant -\sum_{j=2N+1}^{KN} \frac{f(j)}{j} - \sum_{j=2N+1}^{KN} \sum_{i=1}^{f(j)} \sum_{J \in D(j,i)} \frac{1}{\|J\| j} \\
&= -\sum_{j=2N+1}^{KN} \frac{f(j)}{j} - \sum_{N < \|J\| \leqslant 2N} \sum_{2N/\|J\| < \|J'\| \leqslant KN/\|J\|} \frac{1}{\|J\|^2 \|J'\|} \\
&= -\sum_{j=2N+1}^{KN} \frac{f(j)}{j} - \sum_{N < n \leqslant 2N} \frac{f(n)}{n^2} \sum_{2N/\|J\| < n \leqslant KN/\|J\|} \frac{f(n)}{n} \\
&= -c_F \log(K/2) + O\left(1/N^{1/d}\right) - \left(c_F \log K + O(1)\right) \sum_{N < n \leqslant 2N} \frac{f(n)}{n^2}.
\end{aligned}$$

For sufficiently large N, we have the following

$$\sum_{N < n \leqslant 2N} \frac{f(n)}{n^2} = \frac{c_F}{2N} + O\left(1/N^{1+1/d}\right) \geqslant \frac{c_F}{2.9N}.$$

Hence, by (4.3),

$$\begin{aligned}
\mathbf{E} \log \delta(C) &\leqslant -c_F \log K + O\left(1/N^{1/d}\right) - c_F^2 \frac{\log K + O(1)}{2.9N} \\
&\leqslant -c_F \log K - c_F^2 \frac{\log K}{3N},
\end{aligned}$$

for sufficiently large N and K. $\square$

5. Normal value of $\delta(C)$

In this section, we estimate the variance of $\delta(C)$ over $C \in \mathcal{C}(T)$, where $\mathcal{C}(T)$ is the set of residue systems C in a number field $F/\mathbb{Q}$ with $S(C) = T$. As in the case of the integers, we can expect $\delta(C) \approx \alpha(C)$ for almost all $C \in \mathcal{C}(T)$. In fact, we can establish the same result for the variance of $\delta(C)$ as in [7].

Theorem 6. *Let T be a set of distinct ideals with minimum norm $N \geqslant 3$. Let α be the common value of $\alpha(C)$ for $C \in \mathcal{C}(T)$. Then,*

$$\frac{1}{W(T)} \sum_{C \in \mathcal{C}(T)} \left|\delta(C) - \alpha\right|^2 \ll \frac{\alpha^2 \log N}{N^2}.$$

Proof. Let $\alpha = \alpha(C)$, $W = W(T)$ and $W_0 = W_0(T)$. By Lemma 4.1,

$$\frac{1}{W} \sum_{C \in \mathcal{C}(T)} \left|\delta(C) - \alpha\right|^2 = \frac{1}{W} \sum_{C \in \mathcal{C}(T)} \left(\delta(C)^2 - \alpha^2\right). \tag{5.1}$$

Put $u = \sum_{I \in T} 1/\|I\|^2$, and also define

$$\ell(m_i, m_j) = \prod_{\substack{I \in T \\ m_i - m_j \in I}} \frac{\|I\| - 1}{\|I\| - 2},$$

where $(W_0, m_1), \dots, (W_0, m_W)$ is an exact covering system in $F/\mathbb{Q}$. By an argument similar to that in the proof of Theorem 7 of [7], we obtain

$$\begin{aligned} \sum_{C \in \mathcal{C}(T)} \delta(C)^2 &= \frac{\alpha^2}{W} \left(1 - u + O\left(\sum_{n \geqslant N} \frac{f(n)}{n^3}\right)\right) \sum_{1 \leqslant i, j \leqslant W} \ell(m_i, m_j) \\ &= \frac{\alpha^2}{W} \left(1 - u + O\left(\frac{1}{N^2}\right)\right) \sum_{1 \leqslant i, j \leqslant W} \ell(m_i, m_j). \end{aligned} \tag{5.2}$$

Let $M(S) = \prod_{I \in S} (\|I\| - 2)$, where S is a finite set of ideals whose norms are $\geqslant 3$, and let $L(S)$ denote the least common multiple of the members of S. Then,

$$\ell(m_i, m_j) = \prod_{\substack{I \in T \\ m_i - m_j \in I}} \left(1 + \frac{1}{\|I\| - 2}\right) = \sum_{\substack{S \subseteq T \\ m_i - m_j \in L(S)}} \frac{1}{M(S)},$$

and thus,

$$\sum_{1 \leqslant i, j \leqslant W} \ell(m_i, m_j) = \sum_{S \subseteq T} \frac{1}{M(S)} \sum_{\substack{1 \leqslant i, j \leqslant W \\ m_i - m_j \in L(S)}} 1 = W^2 \sum_{S \subseteq T} \frac{1}{M(S)\|L(S)\|}. \tag{5.3}$$

First, considering the case when $\#S \leqslant 1$, we have

$$\sum_{\substack{S \subseteq T \\ \#S \leqslant 1}} \frac{1}{M(S)\|L(S)\|} = 1 + \sum_{I \in T} \frac{1}{(\|I\| - 2)\|I\|} = 1 + u + O\left(1/N^2\right). \tag{5.4}$$

On the other hand, if $S \subseteq T$ and $\#S \geqslant 2$, let J_1, J_2 be two members of S such that for $I \in S$, $\|J_1\| \geqslant \|J_2\| \geqslant \|I\|$. Then $\|L(S)\| \geqslant \|\mathrm{lcm}[J_1, J_2]\| = \|J_1\|\|J_2\|/\|\gcd(J_1, J_2)\|$, so that

$$\begin{aligned} E &:= \sum_{\substack{S \subseteq T \\ \#S \geqslant 2}} \frac{1}{M(S)\|L(S)\|} \\ &\leqslant \sum_{\|J_1\| \geqslant \|J_2\| \geqslant N} \frac{\|\gcd(J_1, J_2)\|}{(\|J_1\| - 2)(\|J_2\| - 2)\|J_1\|\|J_2\|} \sum_{U \subseteq \{I:\ N \leqslant \|I\| \leqslant \|J_2\|\}} \frac{1}{M(U)}. \end{aligned}$$

Since the inner sum is equal to

$$\prod_{N\leqslant\|I\|\leqslant\|J_2\|}\left(1+\frac{1}{\|I\|-2}\right)=\prod_{N\leqslant n\leqslant\|J_2\|}\left(1+\frac{1}{n-2}\right)^{f(n)}$$
$$=\exp\left(\sum_{N\leqslant n\leqslant\|J_2\|}f(n)\log\left(1+\frac{1}{n-2}\right)\right)$$
$$\leqslant\exp\left(\sum_{N\leqslant n\leqslant\|J_2\|}\frac{f(n)}{n-2}\right)\ll\frac{\|J_2\|}{N},$$

by Proposition 1, we have

$$\begin{aligned}E&\ll\frac{1}{N}\sum_{\|J_1\|\geqslant\|J_2\|\geqslant N}\frac{\|\gcd(\|J_1\|,\|J_2\|)\|}{\|J_1\|^2\|J_2\|}\leqslant\frac{1}{N}\sum_{\|J\|\geqslant 1}\sum_{\substack{\|J_1\|\geqslant\|J_2\|\geqslant N\\ J|J_1,\,J|J_2}}\frac{\|J\|}{\|J_1\|^2\|J_2\|}\\
&=\frac{1}{N}\sum_{\|J\|\geqslant 1}\sum_{\|V\|\geqslant\|V'\|\geqslant N/\|J\|}\frac{1}{\|V\|^2\|V'\|\|J\|^2}=\frac{1}{N}\sum_{\|J\|\geqslant 1}\sum_{\|V'\|\geqslant N/\|J\|}\frac{1}{\|V'\|\|J\|^2}\sum_{\|V\|\geqslant\|V'\|}\frac{1}{\|V\|^2}\\
&\ll\frac{1}{N}\sum_{\|J\|\geqslant 1}\sum_{\|V'\|\geqslant N/\|J\|}\frac{1}{\|V'\|^2\|J\|^2}\ll\frac{1}{N}\left(\sum_{\|J\|\leqslant N}\sum_{\|V'\|\geqslant N/\|J\|}\frac{1}{\|V'\|^2\|J\|^2}+\sum_{\|J\|>N}\frac{1}{\|J\|^2}\right)\\
&\ll\frac{1}{N^2}\left(\sum_{\|J\|\leqslant N}\frac{1}{\|J\|}+1\right)\ll\frac{\log N}{N^2}.\end{aligned}\tag{5.5}$$

Combining (5.4) and (5.5), and using (5.3), we obtain

$$\sum_{1\leqslant i,j\leqslant W}\ell(m_i,m_j)=W^2\bigl(1+u+O\bigl((\log N)/N^2\bigr)\bigr).$$

Hence, from (5.2) and $u\ll_F 1/N$, we have

$$\sum_{C\in\mathcal{C}(T)}\delta(C)^2=\alpha^2W\bigl(1+O\bigl((\log N)/N^2\bigr)\bigr).$$

By (5.1), we complete the proof. □

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